This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TPWRS.2018.2811707, IEEE Transactions on Power Systems

A Cumulant-Tensor Based Probabilistic Load Flow Method

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***Abstract*—Probabilistic Load Flow (PLF) analysis is an important part of grid design, optimization and operation due to the uncertainties in the power network for both generation and demand, increasingly so for newly integrated technologies including wind power and plug-in vehicles. A reliable, fast and robust mathematical method for such analyses is a key requirement to help support widespread integration of these new generation and load sources. Conventional deterministic Monte Carlo (MC) analyses, though simple in implementation, becomes too slow as networks become more complex. In this paper, a new cumulant-tensor based method is used to assess power flows. Probability Distribution Functions (PDFs) and reliability indices are generated as final outputs. Furthermore, general correlation between input random variables is included in the analysis. An illustrative 2-bus network is presented along 24-bus IEEE system as case studies, showing the capabilities and increased reliability of the method.**

I. Introduction

POWER system reliability assessments are becoming ever more critical with changes in utilities from vertically integrated entities to more decentralized operations, combined with increasing incorporation of un-controlled distributed generation from renewables and combined cycle plants [1]. Moreover, introduction of electric vehicle charging may change the Probability Density Function (PDF) shape of loads on the network buses from Gaussian to more complicated and general forms [2]. Useful practical indices, such as the probability of voltage limit violations at buses, thermal overloading of power lines, insufficient reactive power etc., are essential parts of long term expansion planing and real-time operation [3]. These quantities must be obtained through probabilistic evaluation of the load flow equations Eq. (1).

Various authors have attempted to address these issues by adopting simplifying assumptions [4], [5]: DC load flow, linearisation of the full non-linear Load Flow (LF) equations, assumed independence of variables, and Normalization. Following such assumptions, one can use various methods to probabilistically solve the load flow problem. A breakdown of previous works and assumptions are categorized in Table I.

In the most elementary approach, MonteCarlo (MC) simulation can be used to acquire an approximation (histogram) for the probability of output variables, however its accuracy is a direct function of the number of stochastic samples taken. This can be very time consuming, particularly for larger systems and when proper statistical convergence is to be achieved.

As an alternative to MC simulations, more direct probabilistic analysis can be used [4]. Linearisation of the LF equations Eq. (1) based on the Taylor expansion around the mean values of the random variables is at the core of such approaches. Various techniques are then deployed to directly obtain probabilistic representations of output variables from the input random quantities: convolution techniques, Fast Fourier Transformations, the method of moments, or cumulant arithmetic. Amongst them, the method of cumulants has been recommended for larger systems with non-normal input variables [6], [7], mostly because it renders an easy-to-implement formulation, particularly in cases with independent inputs Random Variables (RVs). The normality assumption can also lead to a much simpler version of the calculations [8]. However, by incorporating higher order cumulant tensors (Section II-E2) it is possible to incorporate quadratic (or higher) models into cumulant formulations [9].

Evidence of correlation between some of the random variables, principally in short-term studies Section II-D, has motivated accounting for dependencies. Full dependency (linear relationship between random variables [10]) non-/monotonically increasing [11], partial correlation assumptions [12], [13] and justifying the existence of correlation between only loads and/or only generations are examples of such efforts [10], [14]. However, there is a need for one unifying framework which can accept advanced form of correlation and also represent the dependency structure of the output quantities.

In this paper, a novel cumulant-based calculation method is presented which captures the non-linearity of the load flow equations, as well as advanced form of dependency between the random variables (input and/or output). It can work with various types of random variable distribution PDFs too. The following sections detailing the method are as follows. First, a mathematical introduction to load flow modeling, quadratic expansion, correlation structures and cumulant calculations is given. Then the novel methodology is explained, followed by case studies which demonstrate the abilities of method in comparison to available alternatives. Finally the conclusion highlights the main aspects of proposed method and outline future research plans.

II. Mathematical Background

A. Load Flow Modeling

In the load flow problem, the exact relationship between complex power and voltage for each bus is represented as [4]:

$$\vec{S_i^*} = P_i - jQ_i = \vec{V_i^*} \sum_{k=i}^{n} \vec{V_k} \vec{Y_{ik}} \tag{1}$$

Here the admittance of the line ik is $\vec{Y_{ik}} = G_{ik} + jB_{ik}$, with B_{ik} the susceptance and G_{ik} the conductance. $\vec{S_i}$, P_i and Q_i are complex power, active power injection and reactive power

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TABLE I
PROBABILISTIC METHODS AND SIMPLIFICATION APPROACHES

Pseudo-Deterministic		**MC**	[1]
PLF approaches	Direct solution	**FFT**	[15], [16]
		Convolution	[4]
		Cumulant	[17]
	Approximate method	**Moment arithmetic**	[18]
		PEM	[19]
	Limit Theory Heuristic	**Interval arithmetic**	[20]
		Fuzzy logic	[21]
Linearity of LF equations		**Linear**	[4]
		Quadratic	[18], [22]–[24]
		Multi-linearisation	[25], [26]
Dependency between RVs		**Independent**	[4]
		Correlation	[7], [19], [22], [27], [28]
		Linear Dependence	[10], [13]
Normality of input RVs		**Non-Normal**	[10], [19], [29], [30]
		Normal	[14], [22], [24], [28], [31]

injections on bus i. Knowing shunt admittance and possible tap transformer ratios, one can derive a formula for line active and reactive power flow as (B'_{ik}=$0.5B_{ik}$):

$$\vec{S}^{*}_{ik} = \vec{V}_i \vec{I}^{*}_{ik} \qquad \vec{I}_{ik} = \vec{Y}_{ik}\left(\vec{V}_i - \vec{V}_k\right) - jB'_{ik}\vec{V}_i \tag{2}$$

Directly observed from the definition (see Eq. (1)), a nonlinear relationship links the variables of the load flow equations.

Importantly, depending on the chosen coordinate system expressing the complex variables, either *rectangular* or *polar*, Eq. (1) is expressed accordingly. In most probabilistic studies, the polar form is used [4], [17], [22].

For most of the buses (known as *PQ* or load buses) complex power injections are defined, and the bus complex voltage (magnitude and phase angle in polar form) must be calculated. Usually in load flow models a few voltage-controlled buses (*PV*) with fixed voltage magnitude and known real power generation exist too. A reference (or *slack*) bus is also chosen to preserve the balance of power which has dependent power injections with fixed voltage magnitude of 1.0 and zero phase angles [4], [17]. Hence voltage angles (δ_i) for all buses except the slack bus, and voltage magnitudes (V_i) for *PQ* buses, have to be calculated via the LF equations; these unknown quantities are known as output variables. Such quantities are not explicitly described as a function of the given input information. Consequently, the load flow problem can not be solved directly [32]. This issue introduces difficulties in deterministic and probabilistic calculations. Section III covers the common problems associated with the inexplicit nature of the power injection formulation, with possible steps to tackle the issue.

Another formulation may be derived for active and reactive power by substituting the rectangular representation $\vec{V_i} = V_{i,re} + jV_{i,im}$ for the complex voltage in Eq. (1). This form has been used in some deterministic load flow computational approaches in the past [18], [27], [33]–[39]. Note that in order to account for the constant voltage magnitude in *PV* buses, an extra equation has to be introduced for the rectangular form ; thus such a coordinate system is best for systems with more *PQ* buses, as well as multi-machine dynamic response calculations [36]. The extra computation time for the added equations (squared voltage magnitude) is compensated by avoiding trigonometric subroutine evaluations, which show up in the implementation of the polar formulation. It has been proven that the convergence of the deterministic load flow calculations only depends on the system itself and not on the chosen coordinates, as the rate of convergence is determined by the system eigenvalues; these are invariant with the linear coordinate transformation [35]. As will be seen in Section III the rectangular form has advantages for the proposed cumulant-based probabilistic method.

B. Multiple Deterministic Solutions of the Power Flow Equations

In deterministic analyses, given different values of *P* and *Q*, two curves (or surfaces) are identified that may intersect with each other at one, two, or no solution points [33], [34], [37]. This is visually demonstrated for a 2-bus example in Section IV. It was shown that the maximum number of solutions is 2^{n-1} for an n–bus system, of which some of them may be inadmissible [33], [37]. For instance, a given value of transmitted complex power in each line may be obtained by two sets of high voltage/low current or low voltage/high current, amongst which the former is typically preferable [37].

There are two important aspects relating to multiple deterministic solutions of the LF. Firstly, any probabilistic solution of the LF equations corresponds to one deterministic solution. Secondly, probabilistic assessments cannot be obtained for inadmissible solutions [33]. Different methods have been suggested to identify the deterministic solutions of the LF equations [38], [40] . Moreover, the rectangular form of the probabilistic equations are used frequently in these studies, and it has been shown that the polar version has some difficulty in converging to the second solution [38]. Existence of multiple solutions for the LF equations is handled in the currently proposed method, and the input variables' range is chosen so that they do not get close to the no-solution range Fig. 1.

C. Quadratic Power Flow and Taylor Expansion

The power flow equations (1), either in rectangular or polar form, are inherently non-linear. To apply probabilistic analysis techniques, the functions must be approximated using a Taylor series expansion [23]. The rectangular form is exactly represented by a second order expansion, whereas a polar quadratic expansion is associated with approximation error. If the input complex loads and generation (control vectors) are $\mathbf{u}$ and nodal complex voltages (state vectors) $\mathbf{x}$, the power flow equations 1 is functionally expressed as:

$$f(\mathbf{x}) = \mathbf{u} \tag{3}$$

Mean values (known as *expected values*) of the control vector (say $\mathbf{u}_0$) are then used to calculate the corresponding values of the state vectors (say $\mathbf{x}_0$) from Eq. (3) as:

$$f(\mathbf{x}_0) = \mathbf{u}_0 \tag{4}$$

Most authors adopt $\mathbf{x}_0$ from Eq. (4) as the expansion reference point for derivative calculations in the Taylor series [4],

[10], [14]–[16], [29]–[31], [41]–[43]. Once the state vector is calculated the line power flows ($\mathbf{z}$) are explicitly obtained as:

$$\mathbf{z} = g\left(\mathbf{x}\right) \tag{5}$$

A similar procedure as explained earlier is used to evaluate Eq. (5) for $\mathbf{x} = \mathbf{x}_0$.

For non-linear transfer functions f, the expected value of output RVs (*i.e.* $E(\mathbf{x})$) have deviations from $\mathbf{x}_0$ obtained from Eq. (4) [4]. MC simulations also prove such deviations [23]. Therefore using $\mathbf{x}_0$ as the expansion's reference does not always yield a satisfactory estimation of the function, specifically to estimate the load flow equations in regions far from mean values. Another reason for such mismatch can be the existence of large covariances between input RVs. Mathematical proof of such deviations is given in Section II-E1.

Consequently, in general, the linear models fail to accurately predict the PDF of the output RVs, chiefly in the tail regions, the prediction of which is vital in decision making processes. For example obtaining a correct estimate of the probability nodal voltages violate their operational limits in the tail region is critical. To deal with such issues two different methods have been applied in the literature: to apply higher fidelity models in lieu of the linear approximation [18], [22], [23], [44]–[46], or to adopt multiple points picked from the domain for linearisation of the LF equations, and properly combining the result afterwards (known as multi-linearisation) [25], [26]. Nevertheless, in the quadratic version, Eq. (3) is expanded around $\mathbf{x}_0$ as:

$$\Delta\mathbf{u} \approx \mathbf{J}\Delta\mathbf{x} + \tfrac{1}{2}\mathbf{H}\,\mathrm{col}\left(\Delta\mathbf{x}\,\Delta\mathbf{x}^T\right) \tag{6}$$

where for a n–dimensional system $\mathbf{J}$ and $\mathbf{H}$ are the Jacobian ($n \times n$) and Hessian ($n \times n^2$) of Eq. (3). Also it is common in probabilistic studies to investigate only the variations of the RVs about their mean values (in p.u.), hence a transformation of $\Delta\mathbf{u} = \mathbf{u} - \mathbf{u}_0$ and $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0$ is used throughout the procedure. The algebraic expression *col* puts all the elements of its argument in a column vector to make the dimensions of the term in parenthesis agreeable with $\mathbf{H}$. Similar procedure can be applied to obtain the quadratic expansions of the line power flow Eqs. (5). As for the power injection Eq. (3), assuming reversibility by considering a limited domain of variation for $\mathbf{x}$, one can quadratically expand the reverse function as well. The form of this expansion is expressed as:

$$\Delta\mathbf{x} \approx \mathbf{A}\Delta\mathbf{u} + \tfrac{1}{2}\mathbf{B}\ \mathrm{col}\left(\Delta\mathbf{u}\Delta\mathbf{u}^T\right) \tag{7}$$

Due to the inexplicit form of Eq. (3), there is no direct way to work out $\mathbf{B}$ from the second derivatives of the transfer function f (see Eq. (1)). One can use a first order approximation from Eq. (6) and introduce its inverse ($\Delta\mathbf{x} \approx \mathbf{J}^{-1}\Delta\mathbf{u}$) in Eq. (6) to work out $\Delta\mathbf{u}\Delta\mathbf{u}^T$ from the quadratic terms in the parenthesis *i.e.* $\Delta\mathbf{x}\,\Delta\mathbf{x}^T$. Multiplying $\mathbf{J}^{-1}$ into both sides and isolating $\Delta\mathbf{x}$ leads to an approximation for the quadratic form:

$$\Delta\mathbf{x} \approx \mathbf{J}^{-1}\Delta\mathbf{u} + \left(-\tfrac{1}{2}\right)\mathbf{J}^{-1}\mathbf{E}\,\mathrm{col}\left(\Delta\mathbf{u}\Delta\mathbf{u}^T\right) \tag{8}$$

in which the rows of the matrix $\mathbf{E}$ in Eq. (8) are defined as:

$$\mathbf{E}^k = \left(\mathbf{J}^{-1}\right)^T\mathbf{H}^k\left(\mathbf{J}^{-1}\right) \tag{9}$$

This procedure, which involves back-solving of the Hessian matrix for the reverse function, has been used by most authors to drive a second order model for the load flow equations [22], [23], [46]. A study was carried out in the current work to adopt the same methodology for cumulant based calculations, and it was observed that this process would introduce some error, which in conjunction with the cumulant method, would not always lead into improvement of the final results. As will be discussed later in Section III, this paper adopts an alternative methodology which directly uses the Jacobian and Hessian matrices of Eq. (3), and solves for the cumulants of the output RVs, therefore avoiding the afore-mentioned steps in Equations (8) and (9).

It is worth mentioning that using a first order model, one can combine both the power injection Eq. (3) and line power flow Eq. (5) into a single linear equation, and directly calculate the line power flows $\mathbf{z}$ from the values of input RVs i.e. $\mathbf{u}$ [4], [41], which significantly reduces the computation time. In quadratic or higher order models though, the median variables $\mathbf{x}$ must be fully calculated first, then introduced in Eq. (5). This explains the need for adequate knowledge of the interdependencies in $\mathbf{x}$. Thus an extra computational step and a large amount of storage for such information is inevitable.

D. Correlation

The dependency between random RVs is commonly quantified with a covariance matrix, which reflects only the dependency between two of the RVs in a multivariate set of random quantities. One way to obtain covariance of RVs is to use multivariate probability distribution functions of RVs, which is not practically available in most cases. Another approach is to use statistical data, if available, for calculation of covariances, and if not to presume bivariate dependencies using rule-based techniques for possible existing correlation [7]. Correlation is the normalized covariance and its elements assume values between -1 and $+1$. This gives the factors known as Pearson's linear correlation coefficients. Although a strong index to measure linear dependences, correlation coefficients do not give enough information, provided the RVs have a non-linear dependency structure. Owing to the unavailability of joint PDFs in practical applications, the multivariate input random vector is described with individual marginal PDFs and a correlation matrix.

In Probabilistic Load Flow (PLF) practice, three major sources of correlations have been investigated [7], [10], [12], [13], [44], [47]: correlation between load/load, generation/generation and generation/load. The focus of this study is to mathematically address the introduction of all possible correlations into the PLF; hence, nothing is presumed for the individual elements of covariance matrix.

1) Correlation between State RVs: State RVs (*i.e.* $\mathbf{x}$) computed from Eq. (3) are correlated, and due to non-linearity of the transfer function have a complicated dependency structure, making correlation coefficients not fully be satisfactory. Despite the first order LF modeling, in the quadratic PLF it is not possible to effectively combine Eq. (3) and Eq. (3). Thus using cumulants, knowledge of such dependencies is essential to

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work out the line power flows separately calculated in Eq. (3). In this study, high order cumulant tensors (see Section II-E2) are used to characterize such dependencies.

E. Cumulant Calculation

Cumulants, or semi-variants, of a RV with known PDF are calculated from the logarithm of the characteristic function of the RV. Expanding the result in a Maclaurin series, cumulants of the PDF are constants of this series expansion [17], [48]. Such a definition directly connects cumulants to moments (the latter are the constants in Maclaurin series expansion of the characteristic function of the PDF), and therefore one can develop simpler relationships between cumulants and moments without the need for the characteristic function [49], [50]. As explained in [8] and later in Section II-E2, cumulants and moments can be defined for a multivariate set of RVs which leads to the definition of cumulant/moment tensors. The diagonal elements of such tensors are univariate cumulants of individual RVs, which are extensively used to derive the PDF of RVs using one of the PDF reconstruction techniques: Gram-Charlier (GC), Cornish-Fisher (CF), Edgeworth series [51] and Maximum Entropy (ME) [17], [30]. The latter is used in this study since it has been proven to work best in PLF applications, and for lower order of cumulants [17].

Univariate cumulants are invariant to a shift of origin. Also, if a random variable is multiplied with a factor, say l, then the cumulant of order r is multiplied by l^r (which is why cumulants are referred to as semi-invariant [49]). Furthermore, for two independent random variables, the cumulant of order r of their summation is the sum of the rth-order cumulants of each RV. For instance, in LF application, using a first order model for Eq. (3), the univariate cumulants of state RVs are calculated given the cumulants of control RV [6], [17], [28], [43], [52] as:

$$\kappa^r_{\Delta x_i} = \sum_j \left(a^j_i\right)^r \kappa^r_{\Delta u_j} \qquad (10)$$

where the a^j_i is the jth element of the ith row in the linearisation matrix $\mathbf{a}$ knowing $\mathbf{a} = \mathbf{J}^{-1}$ from Eq. (6) , and $\kappa^r_{\Delta x_i}$ and $\kappa^r_{\Delta u_j}$ are the r-th order cumulants of variations of state vector $\mathbf{x}$ and control vector $\mathbf{u}$ respectively. Since the variations are supposed to be around the mean values, the first cumulants of Δu_j and hence Δx_i are zero.

Some authors have attempted to incorporate covariance into cumulant calculations [7], [29], [30], [53]. However, all previous efforts have used linear models for the load flow equations. Therefore, the contribution of utilizing high order cumulant tensors in calculations presented in the current work is novel and provides a technique for calculations of a complete set of cumulants, capable of capturing the non-lineariarities of the governing equations.

1) General Cumulant Calculation: The intention of the current work was to show the effect of using a non-linear model instead of the conventional linear form. However, the cumulant calculations of non-linear functions cannot be summarized in the simple arithmetic form of Eq. (10). Many researches have tried to address this issue, yet they all came no closer than expressions for cumulants of polynomial functions [8], [54], [55]. A more general form can be found in McCullugh's *Tensor Methods in Statistics* [8] and [48]. However, James and Mayne [9] simplified the expression for cumulants of polynomial functions in a more concise form, which is utilized in this paper. Let $\boldsymbol{\psi}$, a random vector of $\{\psi^a\}$, be a polynomial function of $\boldsymbol{\chi}$, a vector of random variables $\{\chi^i\}$ expressed in Einstein notation as:

$$\psi^a = \alpha^a + \beta^a_i \chi^i + \gamma^a_{ij}\chi^i\chi^j + \cdots \quad \left(\gamma^a_{ij} = \gamma^a_{ji}\text{ , etc.}\right) \quad (11)$$

Without loosing generality, assuming all the first order cumulants κ^i to be zero, the joint cumulants $\kappa^{a\ldots b}$ of such a function, up to a certain order, can be explicitly expressed in terms of the input cumulant tensors of $\kappa^{i\ldots j}$. Complete expansion of these expressions are given in [9]. In Appendix VII-A a truncated version of these formulas are given for cumulant tensors up to order 4 for first four cumulants of output RVs.

All the different cumulant procedures in probabilistic analyses can be derived from these equations as special cases, namely the linear model [7], models with independent [6] or dependent RVs [30], [53], or those with full normality assumption of loads [18].

The intent here is to extend the power flow model up to second order terms; for this, the constants α^a, β^a_i and γ^a_{ij} of Eq. (11) are derived from a Taylor series expansion of the general power flow equation for variations of state and control RVs as explained in Eq. (7). Using such variations, as explained in Section II-C, ensures all the first order cumulants of κ^i cancel, and forces all the α^a to be zero in Eq. (11). To capture the non-linearity, the Taylor series are truncated after quadratic terms. Including more terms in the series requires more terms from the cumulant expansion formulas to be included in the calculations of output cumulants, which may lead to longer compution times. However, as was pointed out in Section II-A, the order of the polynomials in the rectangular form of the LF equation is by definition limited to two. Hence, the quadratic polynomials should give a good approximation of the LF equations even in the polar form, and of course a full representation in the rectangular form.

Considering the above, the general form of the cumulant expansion can be truncated to reduce the time of calculation and ease the programming effort required, without dramatically increasing the associated error. Several test cases have been used to examine the effect of such terms in the precision of the computations, and eventually a compact from of Eq. (15) was adopted as the guideline coming out of this study as given in Appendix VII-A.

Interesting insight can be gleaned from Eq. (15). For instance, adopting a first order model eliminates the quadratic terms γ^a_{ij} (or higher) in the expression for κ^a and cancels the contribution of covariance terms κ^{ij}. This highlights the fact that using $\mathbf{x}_0$ as the mean value of state vector $\mathbf{x}$, as is usual in linear PLF, introduces an error in the case of non-linear transfer functions and/or with significant covariances. Thus, relying on this value as the reference for the Taylor expansion to obtain elements of the Jacobian and Hessian as pointed out in Section II-C may introduce error into the whole process. To avoid this, an iterative solver is introduced to update the mean values of the output random variables using the term $\gamma^a_{ij}\kappa^{ij}$. The difference

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between the mean values of the output RVs and the function evaluations of the input means has only been investigated in a few references [18], [23].

2) Cumulant Tensor/Higher Order Cross Cumulants: One of the main contributions of this work is the introduction of cumulant tensors as generalizations of the covariance matrix. As explained in [8], multivariate cross cumulants/moments can be defined in tensor notation for multivariate analysis. Cumulant tensors give important information about the dependency structure between χ^{ij} [8]. For instance, κ^{ij} is a 2-D cumulant tensor equivalent to the covariance of input random variables χ^i and χ^j in Eq. (15), and similarly κ^{jkl} is the third order cumulant tensor, elements of which returns the multivariate cross-cumulants of random vector $\boldsymbol{\chi} = \{\chi^j\}$. Higher order cumulant tensors, as explained in [8], describe more complicated characteristics of the dependence structure between random variables. From a mathematical point of view, if there exists a strong correlation between RVs, the elements of each cumulant tensor are as valuable as the diagonal ones. As such, generally speaking off-diagonal elements in tensor summations such as those in Eq. (15) should not be excluded from the calculations. This however demands an accurate calculation of the cross-cumulant elements. Various techniques and simplifications can be adopted to acquire an estimation for multivariate cross-cumulant terms in cumulant tensors using joint PDF [8], [49] or raw data [8], [49], [56].

Cumulant tensors of order higher than 2 are rarely used in probabilistic analysis, and only their diagonal terms (univariate cumulants) are broadly discussed [6], [7], [14], [17], [28]–[31]. The general cumulant calculation Eq. (15) gives the opportunity to directly calculate all the permutations of $\kappa^{a\ldots b}$ for different output Random Variables (RVs), and generate aforementioned high order cumulant tensors. This information is easily introduced in the next step of practical calculations, such as evaluation of reactive nodal injected powers in PV buses, active/reactive powers in slack bus, or line power flows Eq. (5). Note that introduction of these tensors paves the way for using a novel method in calculating the cumulants of voltage angles and magnitudes, without using back solving procedures, and hence to achieve a lower error especially in tail regions. Unfortunately, this necessitates a large memory usage and also extra nested loops, which may drastically increase the time of calculations, particularly for large systems.

III. Methodology of Cumulant Calculation in Quadratic Load Flow Functions

The main contribution of this work is to blend the general cumulant calculation explained in Section II-E1 and Appendix VII-A with an exact quadratic form of the load flow equations extended in Section II-C. The rectangular form of the nodal complex voltage is used to define the power injection equations (1) in Section II-A to avoid series truncation error, and extra step to separately calculate the Jacobian and Hessian of the LF equations.

Contrary to the polar form of load flow equations, Taylor expansion of quadratic form is unique, hence the curvature (non-linearity) of the LF hyper-surfaces is fully preserved. This means regardless of the distance from the expansion reference point, rectangular form of LF equations is exactly represented with quadratic Taylor series. Hence the error associated with polynomial approximation of LF equations is minimal in the tail regions. As stated in Section II-C such an error becomes significant using linear expansion of polar form, in the regions far from the linearisation point. That being the case, multi-linearisation is suggested in some literatures to reduce tail errors [25], [26]. Adopting the quadratic polynomials obtained from rectangular form of LF equations in the calculations of density functions eliminates the truncation errors of series expansions. It also helps reducing the errors associated with the linearisation reference of the polar LFs equations.

Recalling the inexplicit form of the power injection equations Eq. (3), two procedures are employed; one utilizes back-solving the Hessian of the inverse quadratic function of Eq. (7) as explained in Equations (8) and (9), based on [22], [23], [46]. Equation (15) are subsequently used to calculate the cumulants of output RVs. This method however is not robust and may fail to retain the PDF of the output quantities precisely, mostly due to the linear assumptions to approximate the quadratic terms of inverse function (see Section II-C).

The other approach is a novel iterative solution of the cumulant tensors using the original quadratic equations of Eq. (6) obtained from expansion of Eq. (1) using the rectangular form of the complex nodal voltage. This approach is developed based on the order of magnitude specified for each contributing term in the general cumulant calculation formulas Eq. (15). In these equations the cumulant tensors of the output RVs (state vector $\mathbf{x}$) are referred to as $\kappa^{i\cdots j}$. Similarly, the cumulant tensors of the input RVs (control vector $\mathbf{u}$) are $\kappa^{a\cdots b}$. The general cumulant calculation formulas in Eq. (15) define a non-linear functional relationship between these two sets of tensors, and should one intend to obtain the cumulants of state RVs, such a non-linear complicated equation must be solved for $\kappa^{i\cdots j}$.

Most of the non-linear terms have insignificant contributions; for instance, the term $2\gamma^a_{ik}\gamma^b_{jl}\kappa^{ij}\kappa^{kl}$ has limited impact compared to the first term $\beta^a_i\beta^b_j\kappa^{ij}$. Using only the linear terms with noticeable impact on the error, primary estimations for all output cross-cumulant terms are acquired. In the next step of calculation, a shift made by the non-linear terms with small orders of magnitude is computed and incorporated back into Eq. (15) to update the estimation of output cross-cumulant terms. This procedure iterates a few times and the cumulant tensors are converged to a reasonable approximation (mismatch of $\left\|\kappa^{ij}\right\| < 1^{-10}$ was achieved with maximum 5 iterations).

Overall calculation can be summarized as follow:

1) Obtain cumulant tensors of $\mathbf{u}$ (the input RVs, *i.e.* known nodal power injections) from statistical data as explained in [56] or using predefined PDFs (*i.e.* $\kappa^{ab}, \kappa^{abc}, \kappa^{abcd}$)
2) Solve deterministic LF equations (see Eq. (4)) to obtain $\mathbf{x}_0$ from $\mathbf{u}_0$)
3) Get a Taylor expansion for the rectangular form of LF equations (*i.e.* f in Eq. (3)) expanded around expected values of state RVs ($\mathbf{x}_0$). Inevitably such series have a maximum order of two for rectangular LF equations.

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4) Estimate initial values for cumulant tensors of $\mathbf{x}$ (*i.e.* $\kappa^{ij}, \kappa^{ijk}, \kappa^{ijkl}$, using terms with only linear coefficients in Eq. (15) of Appendix VII-A).
5) Calculate the contributions of the quadratic terms in Eq. (15) of Appendix VII-A.
6) Subtract the result of 5 from the cumulant tensors of $\mathbf{u}$ (*i.e.* $\kappa^{ab}, \kappa^{abc}, \kappa^{abcd}$), which were initially determined in step 1.
7) Recalculate the cumulant tensors of $\mathbf{x}$ (*i.e.* $\kappa^{ij}, \kappa^{ijk}, \kappa^{ijkl}$) from the result of 6 using the terms with only linear coefficients.
8) Go back to step 5 and repeat until an acceptable convergence is occurred (for instance $\left\|\kappa^{ij}\right\| < 1^{-10}$).
9) Reconstruct the PDFs of state RVs (*i.e.* $\mathbf{x}$) using univariate cumulants from the converged cumulant tensors from 7 (*i.e.* $\kappa^{ij}, \kappa^{ijk}, \kappa^{ijkl}$), applying ME [17], [30]

Once a full knowledge of the cumulant tensors of $\mathbf{x}$ is obtained, they can be incorporated back into the power injection equations Eq. (6) to compute the unknown nodal reactive powers and slack bus complex power injection, or similarly be utilized in the line power flow Eq. (5) for calculation of active and reactive power flows between buses. In all cases Eq. (15) of Appendix VII-A is used to calculate the cumulant tensors of non-linear functions, and the ME is applied to reconstruct the PDF from the cumulants of each variable.

IV. Case Study

Two case studies are used to clearly illustrate the viability of the quadratic PLF method explained in Section III; a 2-bus simple system from [27] and a 24-bus IEEE test system from [57]. Though being basic and simple, the 2-bus system has been studied in a number of studies [27], [37], [44]. It clearly provides the platform to explore the existence of multiple solutions. Moreover important factors such as non-linearity, or the effect of correlation and joint cumulant tensors on calculations can be studied in such a 2-bus system. Extending the size of the system adds to the complexity of computations and obscures insight into the method itself. Yet to ensure the abilities of proposed method in realistic cases, a 24-bus IEEE test systems are also studied. In both examples the Monte Carlo and the conventional linear approach are also followed for comparison purposes to depicts the enhancement afforded by the quadratic method.

2-bus case: Equation (12) gives the rectangular form of the LF equations of a 2-bus system, which has a load *PQ* bus ($i = 2$) and a slack bus ($i = 1$); knowing $\vec{Y_{12}} = 1 - j\,2$, one can formulate the active and reactive power on the load *PQ* bus in the rectangular form of the LF equations as (for slack bus $\vec{V_i} = 1 + j\,0$):

$$\begin{aligned} P_2 &= V_{2,re}^2 + V_{2,im}^2 - V_{2,re} + 2V_{2,im} \\ Q_2 &= 2V_{2,re}^2 + 2V_{2,im}^2 - 2V_{2,re} - V_{2,im} \end{aligned} \tag{12}$$

Eliminating the variables $V_{2,re}$ and $V_{2,im}$, and substituting the rectangular relationship of $V_2^2 = V_{2,re}^2 + V_{2,im}^2$ an expression for the determinant of the Jacobian of Eq. (12) is obtained as:

$$16P_2^2 - 16P_2Q_2 + 4Q_2^2 - 20P_2 - 40Q_2 + d^2 - 25 = 0 \tag{13}$$

TABLE II
Load Flow Probabilistic Parameters

	Mean	SD	Probabilist Description
P_2	1.0	0.300	Normal Dist. ($\mu = 1.0$, $\sigma = 0.288$)
Q_2	0.9417	0.1397	Weibull Dist. (Scale = 1, Shape = 8)

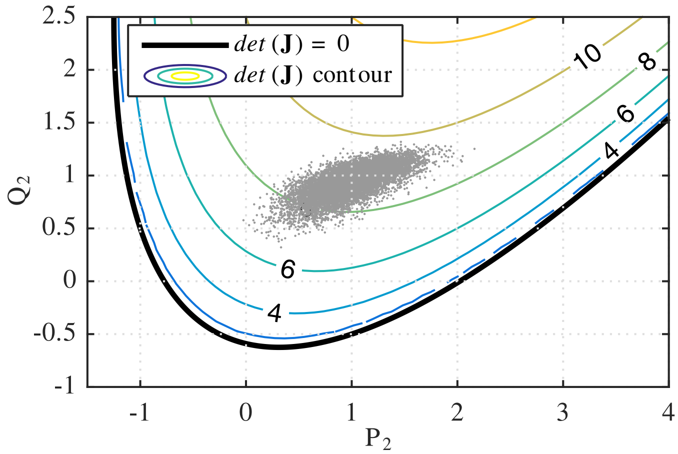

Fig. 1. Contour of $det\,(\mathbf{J})$ for 2-bus system

where $d = \det(\mathbf{J})$. Contours of (13) are plotted in Fig. 1. As is obvious from the plot, for some combinations of P_2 and Q_2 the determinant of the Jacobian does not yield a real value, which means the LF function of Eq. (12) has no real solution. On the line of $d = 0$ there is only one possible solution, while inside the curve two different ones can be obtained separately. In probabilistic calculations, for the assessment to stay valid, the variation of input random variables (here P_2 and Q_2) must not cross the boundary of the no-solution regime [33]. As such, the mean and standard deviation of the input variables in this study have been chosen as stated in Table II. Considering a goal of this paper is also to include non-normalities in input RVs, a Gamma distribution function is assumed for Q_2. Correlation coefficients (ρ) of 0.5, 0.7 and 0.9 between input quantities are chosen for comparison. Also a baseline MC simulation is carried out for 5×10^5 input sequences, and sampled according to the assumptions in Table II. The scatter of such samples is shown in Fig. 1, ensuring input RVs do not enter the no-solution regime. In PLF practice, it is usually important to check whether the voltage magnitude of buses violate their limits, which can be investigated using their PDF. In rectangular form though, as stated in Section II-A, two extra relationship are implemented to obtain the PDF of nodal voltage magnitude and phase angle as functions of real and imaginary components of complex voltage:

$$|V| = \sqrt{V_r^2 + V_i^2} \quad \text{and} \quad \delta = \arctan(\frac{V_i}{V_r}) \tag{14}$$

Such an extra step can be carried out easily, yet computational error involved in linearisation of each of these non-linear equations may slightly shift the result. This is however not the case in calculations of line power flows (see Eq. (5)), considering only the rectangular terms of the nodal complex voltage are involved. Thus error of calculation stays minimal

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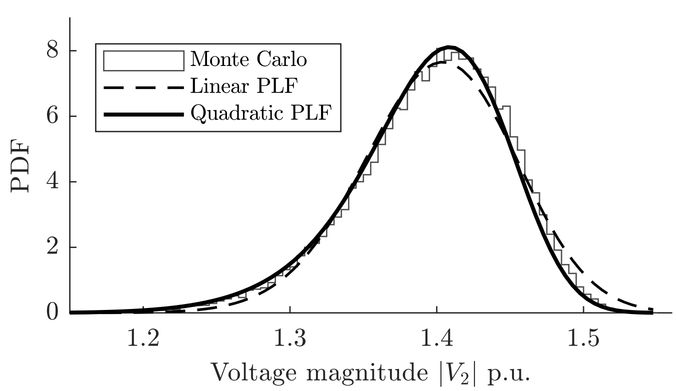

Fig. 2. PDF of Voltage magnitude $|V_2|$

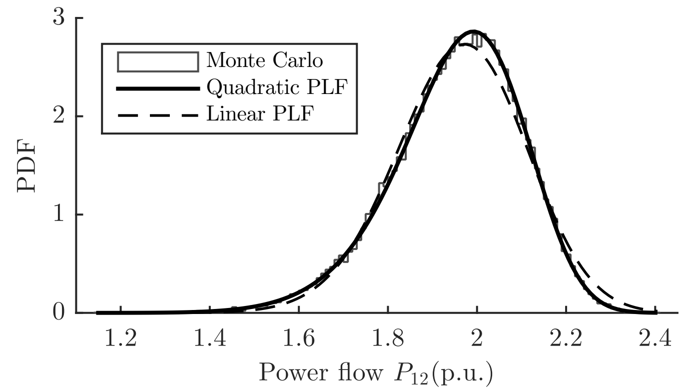

Fig. 3. PDF of line power flow $|P_{12}|$

(see Eq. (2)).

In order to mathematically evaluate the accuracy of calculated PDFs, the Average Root Mean Square (ARMS) index is used, in which the standard deviation of the error between modeled PDF and the one obtained in the numerical MC simulations [17] is computed. This index is calculated for both the proposed quadratic PLF, and its conventional linear alternative. ARMS of both voltage magnitude of bus 2, and the power flow between buses 1 and 2, for a range of correlations is summarized in Table III. Reconstructed PDFs of the output random variables are also presented from both quadratic and linear approaches, plotted in Fig. 2 for voltage magnitudes and in Figure 3 for the power flow between bus 1 and 2, against the histograms of data obtained from Monte Carlo simulations. These plots are used to examine viability of the proposed method in predicting the tail region probabilities in comparison with the traditional linear model.

TABLE III
QUADRATIC AND LINEAR MODEL COMPARISON - ARMS

Correlation coefficient	$\|V_2\|$		$\|P_{12}\|$	
	Lin.	Quad.	Lin.	Quad.
0.5	0.248	0.243	0.072	0.022
0.7	0.089	0.077	0.014	0.005
0.9	0.092	0.061	0.015	0.004

TABLE IV
5% AND 95% CONFIDENCE LEVELS COMPARISON (CORR = 0.7)

$\mathbf{P} \leq$	$\|V_2\|$			$\|P_{12}\|$		
	Lin.	Quad.	MC	Lin.	Quad.	MC
0.05	1.310	1.294	1.298	1.708	1.687	1.686
0.95	1.485	1.469	1.474	2.198	2.171	2.172

Qualitatively speaking, from Fig. 2 and Fig. 3, the proposed quadratic PLF method in conjunction with general cumulant calculations successfully predicts the non-linearity of state equations (3,5), particularly in the tail regions. The confidence levels of 5% and 95% are also given in Table IV, in which the quadratic PLF is associated with smaller errors. In addition, from Table III the new technique is seem to be more reliable in cases with larger correlations compared to linear PLF. One explanation for this trend can be attributed to the involvement of more terms with high order cross cumulant in the quadratic PLF. Larger correlations increase the relative weight of high order expansion coefficients as well as high order cumulant tensors in the final summations.

24-bus IEEE test system: This example attempts to illustrate the contribution of non-linearity in probabilistic studies. The IEEE 24-bus has widely been used in reliability assessments and probabilistic studies [57]. A baseline case was assumed in which the input RVs were defined using uncorrelated normal distributions. Mean values for power injections were set based on nominal operation of the system as suggested in [57], and their standard deviations were assumed to be 25% of their mean values. As expected the reconstructed PDFs for the state variables using a linear series expansion returned perfectly normal distributions. This can be proved mathematically, using Eq. (15) of Appendix VII-A after eliminating all the terms with quadratic coefficients, along with the normality assumption for input RVs. The same steps were repeated using the methodology proposed in this paper using second order expansion of LF equations. The result is demonstrated for voltage magnitude (in *p.u.*) of bus-24 (chosen as one of the worst match between MC and 2nd order expansion), where the plot for quadratic LF exhibits noticeable improvement in predicting the variation of voltage magnitude (see Fig. 4). Applying a linear PLF in conjunction with normal distributions fails to capture any high order cumulants and therefore the results have zero skewness and kurtosis. Including quadratic coefficients in the calculations allows for a reasonable approximation of higher order cumulants and moments.

Table V summarizes the ARMS values calculated for PDFs of voltage magnitudes obtained for both linear and quadratic methods. Evidently the proposed method based on quadratic expansions and cumulant tensors leads to a more accurate approximation of the distributions, hence the uncertainties or variations of the important quantities are calculated with acceptable errors.

Since only normal inputs were used in this example, the non-linearities/non-normality of the outputs came from the grid characteristics. For inputs with non-normal distributions such as wind power injections or electric vehicles charging demands, the outputs are expected to have larger non-normality (*e.g.* non-

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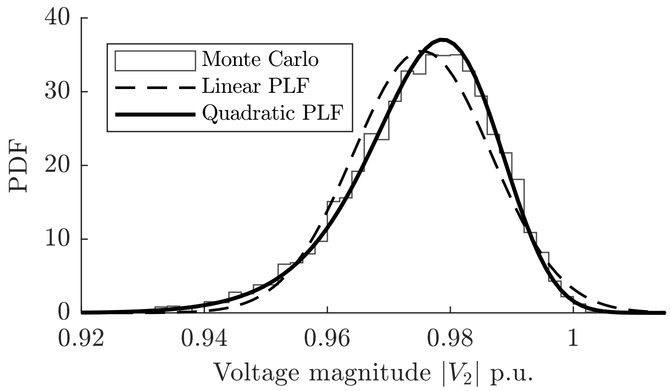

Fig. 4. PDF of Voltage magnitude $|V_2|$ for bus-24 in IEEE24

zero skewness and kurtosis), hence using linear expansions of LF equations would exacerbate the errors. In such cases it would be more important to use the quadratic expansions of LF equations, in order to capture most of the non-linearity/non-normality and reduce the errors.

TABLE V
QUADRATIC AND LINEAR ARMS COMPARISON FOR PDFS OF VOLTAGE MAGNITUDES IN 24-BUS IEEE SYSTEM

Bus	Lin.	Quad.	Bus	Lin.	Quad.
3	1.0452	0.7149	11	5.9318	2.1018
4	1.5034	1.1636	12	3.9528	1.5851
5	2.1308	1.4536	17	47.7456	25.0820
6	1.1001	0.8567	19	6.1047	3.4197
8	1.8401	0.8267	20	11.6136	6.2195
9	2.1493	1.4335	24	2.0081	1.8667
10	1.8384	1.0164			

V. CONCLUSION

This paper considers a new probabilistic method which uses tensor cumulant calculations to retain the non-linearity of the load flow equations in the power network. The method involves an iterative solver to calculate the cumulant tensors of state random variables from those of input quantities. The enhanced accuracy of such a technique is demonstrated for two examples, making the method a reliable candidate for practical applications, particularly those requiring precision in prediction of tail regions of the output PDFs. The results are compared to a baseline Monte Carlo simulation, as well as conventional PLF techniques working only with the linear form of the LF equations. Improvement in prediction of the output random variables' PDF, voltage angles/magnitude and line power flow is achieved through the proposed method. A drawback of such an approach may be the complexity of the implementation of long expansions of cumulant tensors. An ongoing project is addressing such issues, together with practical system analysis applications of the method.

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VII. Appendices

A. General Cumulant Calculation of Polynomial

For a polynomial transfer function such as Eq. (11) the tensors of cumulant for both state ($\mathbf{x}$) and control ($\mathbf{u}$) RVs are connected through a set of summation series as explained extensively in a paper by James and Mayne [9], in which expressions of cumulants up to a certain error for polynomials with maximum order of three is given. Applying the rationalizations summarized in Section II-E1 in terms of truncating the variance matrices, streamlining the input random variable cumulants and simplifying the permutations, yields a much reduced expression for the output cumulant tensors. Equation (15) shows the expressions for the first four cumulants which are used in the current study.

$$\begin{aligned}
\kappa^{a} &= {}_{0}\{\alpha^{a}\} + {}_{1}\{\beta^{a}_{ij}\kappa^{ij}\} + \mathcal{O}\left(\nu^{-5}\right)\\
\kappa^{ab} &= {}_{1}\{\beta^{a}_{i}\beta^{b}_{j}\kappa^{ij}\}\\
&\quad + {}_{2}\left\{\sum^{2}\gamma^{a}_{ij}\beta^{b}_{k}\kappa^{ijk} + \left(2\gamma^{a}_{ik}\gamma^{b}_{jl}\right)\kappa^{ij}\kappa^{kl}\right\}\\
&\quad + {}_{3}\{\left(\gamma^{a}_{ij}\gamma^{b}_{kl}\right)\kappa^{ijkl}\} + \mathcal{O}\left(\nu^{-5}\right)\\
\kappa^{abc} &= {}_{2}\left\{\beta^{a}_{i}\beta^{b}_{j}\beta^{c}_{k}\kappa^{ijk} + 2\sum^{3}\gamma^{a}_{ik}\beta^{b}_{j}\beta^{c}_{l}\kappa^{ij}\kappa^{kl}\right\}\\
&\quad + {}_{3}\left\{\sum^{3}\gamma^{a}_{ij}\beta^{b}_{k}\beta^{c}_{l}\kappa^{ijkl} + \left(2\sum^{6}\gamma^{a}_{ij}\gamma^{b}_{kl}\gamma^{c}_{m}\right.\right.\\
&\qquad \left.\left. + 4\sum^{3}\gamma^{a}_{il}\gamma^{b}_{jm}\beta^{c}_{k}\kappa^{ijk}\kappa^{lm} + 8\gamma^{a}_{ik}\gamma^{b}_{jm}\gamma^{c}_{ln}\right)\kappa^{ij}\kappa^{kl}\kappa^{mn}\right\}\\
\kappa^{abcd} &= {}_{3}\left\{\beta^{a}_{i}\beta^{b}_{j}\beta^{c}_{k}\beta^{d}_{l}\kappa^{ijkl} + 2\sum^{12}\gamma^{a}_{il}\beta^{b}_{j}\beta^{c}_{k}\beta^{d}_{m}\kappa^{ijk}\kappa^{lm} + \right.\\
&\quad \left. + 4\sum^{12}\gamma^{a}_{ik}\gamma^{b}_{jm}\beta^{c}_{l}\beta^{d}_{n}\kappa^{ij}\kappa^{kl}\kappa^{mn}\right\} + \mathcal{O}\left(\nu^{-5}\right) \qquad (15)
\end{aligned}$$

The summation signs refer to permutations of $a, b, c...$ which produce distinct terms in the full expression of cumulant tensors for $\{y^{a}\dots y^{b}\}$, where $\mathcal{O}\left(\nu^{r}\right)$ is the order of magnitude of error and ν^{r} refers to ${}_{r}\{\}$. A more complete exposition can be found in [9].